

STIT Tessellations – Ergodic Limit Theorems and Bounds for the Speed of Convergence

Servet Martínez

Departamento Ingeniería Matemática and Centro Modelamiento Matemático,
Universidad de Chile,
UMI 2807 CNRS, Casilla 170-3, Correo 3, Santiago, Chile.
Email: smartine@dim.uchile.cl

Werner Nagel

Friedrich-Schiller-Universität Jena,
Institut für Stochastik,
Ernst-Abbe-Platz 2, D-07743 Jena, Germany.
Email: werner.nagel@uni-jena.de

Abstract

We consider homogeneous STIT tessellations in the ℓ -dimensional Euclidean space \mathbb{R}^ℓ . Based on results for the spatial β -mixing coefficient an upper bound for the variance of additive functionals of tessellations is derived, using results by Yoshihara and Heinrich. Moreover, ergodic theorems are applied to subadditive functionals.

Keywords: Stochastic geometry; Random process of tessellations; STIT tessellation; Ergodic theory; β -mixing

AMS subject classification: 60D05, 60J25, 60J75, 37A25

1 Introduction

Random tessellations form an important class of models studied in stochastic geometry. Besides the well-established Poisson hyperplane tessellations and Poisson-Voronoi tessellations, the STIT tessellation extends the variety of models that are interesting from a theoretical as well as a practical point of view. An essential aspect for such models is whether ergodic limit theorems or even CLT's can be shown for certain functionals of tessellations. This forms the background for statistical applications, as parameter estimation, tests or confidence intervals.

For the Poisson-Voronoi tessellations, a seminal paper dealing with these aspects is [2]. There is also an appropriate concept developed for mixing coefficients in space, including sufficient conditions for CLT's. A recent review can be found in [3].

For STIT tessellations, which were introduced in [8], a profound study of second order properties and the limit behavior of certain functionals was provided in [12, 14]. These papers also contain a good intuitive interpretation and

a discussion of the results. It becomes evident that STIT tessellations show a long-distance (in space) dependency behavior which is in clear contrast to Poisson-Voronoi tessellations. This causes a quite different limiting behavior of STIT compared to Poisson-Voronoi tessellations.

In [12, 14] detailed results for the asymptotic variances (in a growing observation window) are proved for the number of vertices of planar stationary and isotropic STIT tessellations and for the total edge length (planar case) or the total surface area/volume of cell boundaries (dimension ≥ 2), respectively. This work is mainly based on an application of martingale theory.

In the present paper we apply our results from [6] on the β -mixing coefficient of STIT tessellations in two ways. Consider a translation invariant functional X defined on tessellations. The convergence is studied of values of X on a growing sequence of bounded observation windows, divided by the respective volumes of these windows. Due to the ergodicity of homogeneous STIT tessellations the limit should be the expectation of X for a window of unit volume.

First, for additive functionals X we use the Yoshihara-Heinrich method derived in [2] (referring back to [19]) in order to find a universal upper bound for the variance of the X -value for a given window. This implies a proposition on the speed of L^2 -convergence of the rescaled X -values for a growing sequence of windows.

Furthermore, we apply results from ergodic theory (Theorem by Akcoglu-Krengel and by Smythe, see [4]) for multi-parameter subadditive processes to homogeneous STIT tessellations. This yields a.s. convergence and L^1 convergence even if X is a subadditive functional.

In Section 2 we give a short description of the STIT tessellation process, as a process in time with values in the set of all tessellations of the ℓ dimensional Euclidean space \mathbb{R}^ℓ . Then, in Section 3, we recall the definition of the β -mixing coefficient in general and for random tessellations in particular, and we quote our result from [6] for STIT tessellations. The main result concerning the variance of additive functionals X is then given in Section 4 together with some examples. The theorem is proved in Section 5. Finally, Section 6 provides the application of ergodic theorems to subadditive functionals X for STIT.

2 STIT tessellations

For the first time, STIT tessellations were defined in [8]. There, a construction of STIT in bounded windows was described in all detail. An alternative but equivalent construction was given in [5]. Throughout the whole paper, we consider tessellations of the ℓ -dimensional Euclidean space \mathbb{R}^ℓ with dimension $\ell \geq 2$. (Note that the STIT tessellation in \mathbb{R}^1 is generated by a stationary Poisson point process on the real axis.)

A detailed and sound definition of the measurable space of tessellations of a Euclidean space is given in [11] (Ch. 10, Random Mosaics). A tessellation is a

set y of polytopes (the cells) with disjoint interiors and covering the Euclidean space, where each bounded subset of \mathbb{R}^ℓ is intersected by only finitely many cells (locally finiteness condition). On the other hand, a tessellation can as well be considered as a closed set $\partial y \subset \mathbb{R}^\ell$ which is the union of the cell boundaries. There is an obvious one-to-one relation between both ways of description of a tessellation, and their measurable structures can be related appropriately, see [11, 5]. Denote by \mathbb{T} the set of all tessellations of \mathbb{R}^ℓ .

Let \mathcal{C} be the set of all compact subsets of \mathbb{R}^ℓ . We endow \mathbb{T} with the Borel σ -algebra $\mathcal{B}(\mathbb{T})$ of the Fell topology (also known as the topology of closed convergence), namely

$$\mathcal{B}(\mathbb{T}) = \sigma(\{\{y \in \mathbb{T} : \partial y \cap C = \emptyset\} : C \in \mathcal{C}\})$$

($\sigma(\mathcal{I})$ is the smallest σ -algebra containing the class \mathcal{I} of sets.)

We can also consider tessellations of a bounded window W which is assumed to be a polytope (i.e. the convex hull of a finite set of points) with nonempty interior. Denote the set of all those tessellations by $\mathbb{T} \wedge W$. If $y \in \mathbb{T}$ we denote by $y \wedge W$ the induced tessellation on W . Its boundary is defined by $\partial(y \wedge W) = (\partial y \cap W) \cup \partial W$.

For a window W we introduce the following sub- σ -algebras of $\mathcal{B}(\mathbb{T})$:

$$\begin{aligned} \mathcal{B}(\mathbb{T}_W) &= \sigma(\{\{y \in \mathbb{T} : \partial y \cap C = \emptyset\} : C \subseteq W, C \in \mathcal{C}\}) \text{ and} \\ \mathcal{B}(\mathbb{T}_{W^c}) &= \sigma(\{\{y \in \mathbb{T} : \partial y \cap C = \emptyset\} : C \subset W^c, C \in \mathcal{C}\}). \end{aligned} \quad (1)$$

We notice that if $W' \subseteq W$ then $\mathcal{B}(\mathbb{T}_{W'}) \subseteq \mathcal{B}(\mathbb{T}_W)$ and $\mathcal{B}(\mathbb{T}_{W^c}) \subseteq \mathcal{B}(\mathbb{T}_{W'^c})$.

We will consider here the action of (spatial) translations on the space of tessellations \mathbb{T} . For $a \in \mathbb{R}^\ell$, the translated tessellation $y + a$ is defined by $\partial(y + a) = (\partial y) + a = \{z + a : z \in \partial y\}$.

2.1 Construction of STIT

Now we describe a construction of the STIT process, which is a particular continuous time Markov process on \mathbb{T} . Let $(\mathcal{H}, \mathfrak{H})$ denote the measurable space of all hyperplanes (i.e. $(\ell - 1)$ -dimensional affine subspaces) in \mathbb{R}^ℓ (cf. [11]) and Λ be a (non-zero) measure on this space of hyperplanes that satisfies:

Assumption I: *The measure Λ on $(\mathcal{H}, \mathfrak{H})$ is translation invariant and possesses the following locally finiteness property: $\Lambda([B]) < \infty$ for all polytopes $B \subset \mathbb{R}^\ell$, where we denote*

$$[B] = \{H \in \mathcal{H} : H \cap B \neq \emptyset\}.$$

Moreover, the support of Λ is such that there is no line in \mathbb{R}^ℓ with the property that all hyperplanes of the support are parallel to it (in order to obtain a.s. bounded cells in the constructed tessellation, cf. [11], Theorem 10.3.2).

This assumption implies $0 < \Lambda([W]) < \infty$ for every window W . We denote by $\Lambda_{[W]}$ the restriction of Λ to $[W]$ and by $\hat{\Lambda}_{[W]} = \Lambda([W])^{-1} \Lambda_{[W]}$ the normalized probability measure.

Let us give a brief description of the construction of $Y \wedge W = (Y_t \wedge W : t \geq 0)$, the random process of STIT tessellations on W . Let $(\tau_n, n \in \mathbb{N})$ be a sequence of independent identically distributed random variables, each one of them exponentially distributed with parameter 1.

Algorithm.

(a) We define $Y_0 \wedge W = \{W\}$ the trivial tessellation for the window W , and its unique cell is denoted by $C^1 = W$.

(b) Any cell C^i which is generated in the course of the construction has the lifetime $\Lambda([C^i])^{-1} \tau_i$, i.e. an exponentially distributed lifetime with parameter $\Lambda([C^i])$. At the end of its lifetime the cell C^i is divided into two new cells by a random hyperplane H_i with law $\hat{\Lambda}_{[C^i]} = \Lambda([C^i])^{-1} \Lambda_{[C^i]}$, where H_i is (if C^i is given) conditionally independent from all τ_j and all H_j , $j < i$.

(c) This procedure is performed for any extant cell independently. It is easy to see that at any time a.s. at most one cell dies and so a.s. at most only two cells are born. We enumerate the cells according to the time t when they are born: if at this time there have already born n cells, then we denote the cells resulted from a division at t by C^{n+1} and C^{n+2} .

With this notation, at each time $t > 0$ the tessellation $Y_t \wedge W$ is constituted by the cells C^i which 'live' at time t . In [8] it was shown that there is no explosion, so at each time $t > 0$ the number of cells of $(Y \wedge W)_t$, denoted by ξ_t , is finite a.s..

By construction, the holding time at t

$$\sigma_t = \inf\{s > 0 : (Y \wedge W)_{t+s} \neq (Y \wedge W)_t\}.$$

is exponentially distributed with parameter

$$\zeta((Y \wedge W)_t) \text{ where } \zeta(y) = \sum_{C \in y} \Lambda([C]) \text{ for } y \in \mathbb{T} \wedge W. \quad (2)$$

On every window W there exists $Y \wedge W = ((Y \wedge W)_t : t > 0)$, which we call a STIT tessellation process. It turns out to be a pure jump Markov process and hence it has the strong Markov property. Furthermore, if W' and W are windows such that $W' \subseteq W$, then for any $t > 0$: $(Y \wedge W)_t \wedge W' \stackrel{D}{=} (Y \wedge W')_t$, where $\stackrel{D}{=}$ denotes the identity of distributions (for a proof see [8]). This yields the existence of a STIT tessellation (Y_t) of \mathbb{R}^ℓ such that for all windows W we have $Y_t \wedge W \stackrel{D}{=} (Y \wedge W)_t$.

A global construction for a STIT process was provided in [7] which shows straightforwardly that $Y = (Y_t : t > 0)$ is a Markov process, so for all $t > 0$, Y_t

takes values in \mathbb{T} . A property we shall use is that Y_t is spatially stationary (also referred to as homogeneous) i.e. its law is translation invariant, $Y_t \stackrel{D}{=} Y_t + a$ for all $a \in \mathbb{R}^\ell$. We will denote by P_t the law of Y_t on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$.

3 An upper bound for the spatial β -mixing coefficient of STIT tessellations

3.1 The β -mixing coefficient

First, we recall the general definition of the β -mixing coefficient for pairs of σ -algebras as given in [17], see also [2]. For a probability space $(\Omega, \mathcal{D}, \mathbf{P})$ and sub- σ -algebras $\mathcal{A}, \mathcal{B} \subseteq \mathcal{D}$ denote by $\mathbf{P}_{\mathcal{A}}, \mathbf{P}_{\mathcal{B}}$ the restrictions of \mathbf{P} to \mathcal{A} and \mathcal{B} respectively, and by $\mathbf{P}_{\mathcal{A} \otimes \mathcal{B}}$ the restriction to $\mathcal{A} \otimes \mathcal{B}$ of the image measure of \mathbf{P} on $\Omega \times \Omega$ induced by the diagonal mapping $\omega \mapsto (\omega, \omega)$. Also, and as usual, we denote by $\mathbf{P}_{\mathcal{A}} \otimes \mathbf{P}_{\mathcal{B}}$ the product measure of $\mathbf{P}_{\mathcal{A}}$ and $\mathbf{P}_{\mathcal{B}}$. Then, the β -mixing coefficient can be given by one of the following expressions:

$$\begin{aligned} \beta(\mathcal{A}, \mathcal{B}) &= \mathbb{E} \left(\sup_{B \in \mathcal{B}} |\mathbf{P}(B|\mathcal{A}) - \mathbf{P}(B)| \right) \\ &= \sup_{C \in \mathcal{A} \otimes \mathcal{B}} |\mathbf{P}_{\mathcal{A} \otimes \mathcal{B}}(C) - (\mathbf{P}_{\mathcal{A}} \otimes \mathbf{P}_{\mathcal{B}})(C)| \\ &= \frac{1}{2} \sup_{(\overline{A}, \overline{B})} \sum_{r=1}^I \sum_{s=1}^J |\mathbf{P}(A_r \cap B_s) - \mathbf{P}(A_r)\mathbf{P}(B_s)|, \end{aligned}$$

where in the last expression the supremum is taken over all pairs of finite partitions of Ω : $\overline{A} = \{A_r, r = 1, \dots, I\}$ and $\overline{B} = \{B_s, s = 1, \dots, J\}$ with $I, J \in \mathbb{N}$, for events $A_r \in \mathcal{A}, B_s \in \mathcal{B}$.

Now we consider the special case of pairs of σ -algebras (1) which are defined with respect to pairs of windows. This is tailored for dealing with models in stochastic geometry (see the seminal paper [2] and also [1]).

In [2] the β -mixing rate for a random tessellation with distribution \mathbf{P} on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ is introduced as follows. Consider a pair of windows $W' = [-a, a]^\ell \subset W = [-b, b]^\ell$, $0 < a < b$. Then define

$$\beta(a, b) = \beta(\mathcal{B}(\mathbb{T}_{W'}), \mathcal{B}(\mathbb{T}_{W^c})), \quad (3)$$

where $\mathcal{B}(\mathbb{T}_{W'})$ and $\mathcal{B}(\mathbb{T}_{W^c})$ are given by (1).

In [6] we proved an upper bound for $\beta(a, b)$ for the STIT tessellation Y_t . To supply it we will introduce the whole context. For a fixed time $t > 0$ we choose $\mathbf{P} = P_t$ on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, the marginal distribution of Y_t , to study $\beta(a, b)$.

Consider the windows $W' = [-a, a]^\ell$, $W = [-b, b]^\ell$ with $0 < a < b$ and denote their $(\ell - 1)$ -dimensional facets by f'_r and f_r respectively, $r = 1, \dots, 2\ell$.

We define them for $r = 1, \dots, \ell$ as

$$f'_r = [-a, a] \times \dots \times [-a, a] \times \{a\} \times [-a, a] \times \dots \times [-a, a]$$

with the singleton $\{a\}$ standing at the r -th position, and $f'_{r+\ell} = -f'_r$ for $r = 1, \dots, \ell$. The f_r are defined as the f'_r respectively, by replacing a by b . Since f'_r and f_r are disjoint closed convex sets, the class of their separating hyperplanes $G_r(a, b) = [f'_r | f_r]$ is nonempty and it is a measurable set, that belongs to \mathfrak{H} . We need the following additional assumption on Λ on the sets of separating hyperplanes $G_r(a, b) = [f'_r | f_r]$, $r = 1, \dots, 2\ell$.

Assumption II: For all $0 < a < b$ and the windows $W' = [-a, a]^\ell$, $W = [-b, b]^\ell$ we assume that $\Lambda(G_r(a, b)) > 0$ for all $r = 1, \dots, 2\ell$.

In [6] we proved (in Theorem 5.3) the following property on the β -mixing coefficient for the STIT tessellations.

Theorem 1 *Let (Y_t) be the STIT tessellation determined by the hyperplane measure Λ satisfying Assumptions I and II. Then for all fixed $t > 0$, and $\mathbf{P} = P_t$ being the distribution of Y_t on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, it holds $\lim_{b \rightarrow \infty} \beta(a, b) = 0$ for all $a > 0$. Moreover, defin for all $\theta \in (0, 1)$ there exists a constant $\chi = \chi(t, a, \theta) < \infty$ such that $\beta(a, b) \leq \chi b^{-\theta}$ for all $b > a > 0$. \square*

4 Main result

Using our results for an upper bound of the β -mixing coefficient, we obtain a proposition concerning L^2 -convergence including a bound for the speed of convergence.

For $c = (c_1, \dots, c_\ell)$, $d = (d_1, \dots, d_\ell) \in \mathbb{R}^\ell$ let $[c, d[= \prod_{r=1}^\ell [c_r, d_r[$. We denote by $\mathcal{V} = \{[c, d[: c, d \in \mathbb{R}^\ell, [c, d[\neq \emptyset\}$ the family of nonempty half-open cuboids.

For $y \in \mathbb{T}$ and $V \in \mathcal{V}$, by $y \wedge V$ we mean the tessellation induced by y in V .

Theorem 2 *Let Y_t be the STIT tessellation at time $t > 0$ determined by the hyperplane measure Λ satisfying Assumptions I and II. Further, for any $V \in \mathcal{V}$ let $X(V, \cdot) : \mathbb{T} \wedge V \rightarrow \mathbb{R}$ be a functional with the following properties (briefly we write $X(V, y)$ for $X(V, y \wedge V)$):*

1. *If $n \in \mathbb{N}$ and $V_1, \dots, V_n \in \mathcal{V}$ are pairwise disjoint such that $\bigcup_{r=1}^n V_r \in \mathcal{V}$, then*

$$X\left(\bigcup_{r=1}^n V_r, \cdot\right) = \sum_{r=1}^n X(V_r, \cdot) \quad (\text{additivity}). \quad (4)$$

2. *$X(V, y) = X(V + i, y + i)$ for all $V \in \mathcal{V}$, $i \in \mathbb{R}^\ell$, $y \in \mathbb{T}$.*
3. *For some $\delta > 0$ and some $V \in \mathcal{V}$ we have $\mathbb{E}(X(V, Y_t)^{2+\delta}) < \infty$.*

Then for all $\theta \in (0, 1)$

$$\text{Var} \left(\frac{1}{(2n)^\ell} X([-n, n]^\ell, Y_t) \right) \leq O \left(n^{-\theta \frac{\delta}{2+\delta}} \right) \quad n \rightarrow \infty. \quad (5)$$

We note that inequality (8) which we will supply in the proof of this theorem, will provide an upper bound for the variance for any fixed $n \in \mathbb{N}$.

Since the law of Y_t is translation invariant we have $Y_t \wedge W \stackrel{D}{=} [(Y_t + a) \wedge (W + a)] - a$ for all $a \in \mathbb{R}^\ell$. Hence,

$$\mathbb{E} (X([-n, n]^\ell, Y_t)) = (2n)^\ell \mathbb{E} (X([0, 1]^\ell, Y_t)),$$

and the variance can be written as

$$\text{Var} \left(\frac{1}{(2n)^\ell} X([-n, n]^\ell, Y_t) \right) = \mathbb{E} \left(\left(\frac{1}{(2n)^\ell} X([-n, n]^\ell, Y_t) - \mathbb{E} X([0, 1]^\ell, Y_t) \right)^2 \right).$$

Therefore relation (5) immediately indicates L^2 -convergence in the law of large numbers. The expectation $\mathbb{E} (X([0, 1]^\ell, Y_t))$ is called the X -density of Y_t .

Let us compare our theorem with the results by Schreiber and Thäle. The papers [12, 13] contain a rich material concerning the second-order properties of STIT tessellations. Here we only refer to some asymptotic results that can be compared immediately with our proposition.

Let Y_t be a spatially stationary *and isotropic* STIT tessellation at time $t > 0$.
(i) Let $\ell = 2$. If the random field $X([-n, n]^2, Y_t)$ is the number of vertices, or $X([-n, n]^2, Y_t)$ is the number of center points of maximal $(I-)$ segments, or $X([-n, n]^2, Y_t)$ is the total length of edges, of Y_t in $[-n, n]^2$, then:

$$\text{Var} \left(\frac{1}{(2n)^2} X([-n, n]^2, Y_t) \right) = O(n^{-2} \ln n) \quad \text{for } n \rightarrow \infty,$$

(see [12], Corollary 2, and [13], Theorem 6.1).

(ii) Let $\ell \geq 3$. If $X([-n, n]^\ell, Y_t)$ is the total surface area of cell boundaries of Y_t in $[-n, n]^\ell$ then:

$$\text{Var} \left(\frac{1}{(2n)^\ell} X([-n, n]^\ell, Y_t) \right) = O(n^{-2}) \quad \text{for } n \rightarrow \infty,$$

(see [13], Theorem 6.1). Hence, in the above particular cases the asymptotic variance is considerably smaller than our upper bound (5).

Some examples where Theorem 2 can be applied and up to now no better bound for the variance is known, are the following ones.

1. As in (i) and (ii) above, but anisotropic STIT tessellations.
2. (1) $\ell \geq 3$, $0 \leq k \leq \ell - 1$:

- (a) total k -volume of k -dimensional faces of cells in $[-n, n]^\ell$,
- (b) number of reference points of k -dimensional faces of cells in $[-n, n]^\ell$,
- (c) number of reference points of other k -dimensional objects of Y_t in $[-n, n]^\ell$ (see [18, 16]).

Reference points are uniquely determined points assigned to all the considered sets by the same rule, e.g. their circumcenters or their 'most left' or their 'lowest' points.

5 The Yoshihara-Heinrich method for upper bounds for the variance

5.1 Upper bound for the covariance

We will obtain the upper bound (5) by using the following result, that can be found in [2], formula (4.6).

Theorem 3 (*Yoshihara-Heinrich*) *Let $(\Omega, \mathcal{D}, \mathbf{P})$ be a probability space, $\mathcal{A}, \mathcal{B} \subseteq \mathcal{D}$ sub- σ -algebras and $\beta(\mathcal{A}, \mathcal{B})$ the respective β -mixing (absolute-regularity) coefficient. Then for any $\mathcal{A} \otimes \mathcal{B}$ -measurable function $h : \Omega \times \Omega \rightarrow \mathbb{R}^d$, $d \geq 1$, and all $\delta > 0$*

$$\begin{aligned} & \int |h| |\mathrm{d}(\mathbf{P}_{\mathcal{A} \otimes \mathcal{B}} - \mathbf{P}_{\mathcal{A}} \otimes \mathbf{P}_{\mathcal{B}})| \\ & \leq 2 \max \left\{ \left(\int |h|^{1+\delta} \mathrm{d}\mathbf{P}_{\mathcal{A} \otimes \mathcal{B}} \right)^{\frac{1}{1+\delta}}, \left(\int |h|^{1+\delta} \mathrm{d}\mathbf{P}_{\mathcal{A}} \otimes \mathbf{P}_{\mathcal{B}} \right)^{\frac{1}{1+\delta}} \right\} (\beta(\mathcal{A}, \mathcal{B}))^{\frac{\delta}{1+\delta}} \end{aligned} \quad (6)$$

We note that in the case $\beta(\mathcal{A}, \mathcal{B}) = 0$ both sides vanish, and when $\beta(\mathcal{A}, \mathcal{B}) > 0$ and $\int |h|^{1+\delta} \mathrm{d}\mathbf{P}_{\mathcal{A}} \otimes \mathbf{P}_{\mathcal{B}} = \infty$ the right hand side is ∞ .

Choosing the function h as $h(\omega_1, \omega_2) = X(\omega_1) \cdot Z(\omega_2)$, $(\omega_1, \omega_2) \in \Omega \times \Omega$, for real-valued random variables X, Z (with finite second moments) yields the following upper bound for their covariance.

Corollary 1 *For all real valued random variables $X, Z \in L^2(P)$ and all $\delta > 0$*

$$|\mathrm{Cov}(X, Z)| \leq 2 \left(\mathbb{E}(|X|^{2+\delta}) \right)^{\frac{1}{2+\delta}} \left(\mathbb{E}(|Z|^{2+\delta}) \right)^{\frac{1}{2+\delta}} (\beta(\sigma(X), \sigma(Z)))^{\frac{\delta}{2+\delta}},$$

where $\sigma(X), \sigma(Z)$ denote the σ -algebras generated by X and Z respectively.

Proof 1 *Theorem 3 yields for $\theta > 0$*

$$\begin{aligned}
& |Cov(X, Z)| \\
&= \left| \int X \cdot Z d\mathbf{P}_{\sigma(X) \otimes \sigma(Z)} - \int X \cdot Z d\mathbf{P}_{\sigma(X)} \otimes \mathbf{P}_{\sigma(Z)} \right| \\
&\leq \int |X \cdot Z| |d(\mathbf{P}_{\sigma(X) \otimes \sigma(Z)} - \mathbf{P}_{\sigma(X)} \otimes \mathbf{P}_{\sigma(Z)})| \\
&\leq 2 \max \left\{ \left(\int |XZ|^{1+\theta} d\mathbf{P}_{\sigma(X) \otimes \sigma(Z)} \right)^{\frac{1}{1+\theta}}, \left(\int |XZ|^{1+\theta} d\mathbf{P}_{\sigma(X)} \otimes \mathbf{P}_{\sigma(Z)} \right)^{\frac{1}{1+\theta}} \right\} \\
&\quad \cdot (\beta(\sigma(X), \sigma(Z)))^{\frac{\theta}{1+\theta}}
\end{aligned}$$

Applying the Hölder inequality we obtain

$$\begin{aligned}
& \left(\int |X|^{1+\theta} |Z|^{1+\theta} d\mathbf{P}_{\sigma(X) \otimes \sigma(Z)} \right)^{\frac{1}{1+\theta}} \\
&\leq \left(\int |X|^{2+2\theta} d\mathbf{P}_{\sigma(X) \otimes \sigma(Z)} \right)^{\frac{1}{2(1+\theta)}} \left(\int |Z|^{2+2\theta} d\mathbf{P}_{\sigma(X) \otimes \sigma(Z)} \right)^{\frac{1}{2(1+\theta)}}.
\end{aligned}$$

On the other hand, the Jensen inequality yields,

$$\begin{aligned}
\left(\int |XZ|^{1+\theta} d\mathbf{P}_{\sigma(X)} \otimes \mathbf{P}_{\sigma(Z)} \right)^{\frac{1}{1+\theta}} &= \left(\int |X|^{1+\theta} d\mathbf{P}_{\sigma(X)} \int |Z|^{1+\theta} d\mathbf{P}_{\sigma(Z)} \right)^{\frac{1}{1+\theta}} \\
&\leq \left(\int |X|^{2+2\theta} d\mathbf{P}_{\sigma(X)} \right)^{\frac{1}{2(1+\theta)}} \left(\int |Z|^{2+2\theta} d\mathbf{P}_{\sigma(Z)} \right)^{\frac{1}{2(1+\theta)}}.
\end{aligned}$$

Now choose $\delta = 2\theta$, and note that

$$\mathbb{E}(|X|^{2+\delta}) = \int |X|^{2+\delta} d\mathbf{P}_{\sigma(X)} \text{ and } \mathbb{E}(|Z|^{2+\delta}) = \int |Z|^{2+\delta} d\mathbf{P}_{\sigma(Z)},$$

and the result is shown.

In the next Section the results for the covariance are applied to provide an upper bound for the variance of additive functionals in large windows.

5.2 Proof of Theorem 2

Consider the half-open cube $W_n^o = [-n, n[^\ell$, $n \in \mathbb{N}$, and partition it into half-open unit cubes

$$c_i := \prod_{r=1}^{\ell} [i_r, i_r + 1[\text{ for } i = (i_1, \dots, i_\ell) \in \mathbb{Z}^\ell \cap [-n, n[^\ell.$$

Now, if X is an additive functional then $X(W_n^o, Y_t) = \sum_i X(c_i, Y_t)$. For simplicity we define $X(c_i, Y_t) = 0$ if $i \notin \mathbb{Z}^\ell \cap [-n, n[^\ell$, and we write X_i for $X(c_i, Y_t)$.

On \mathbb{Z}^ℓ we use the maximum metric

$$d(i, j) = \max\{|i_r - j_r| : r = 1, \dots, \ell\}, \quad i, j \in \mathbb{Z}^\ell.$$

Note that the number of points with integer coordinates in the ball of radius $k > 0$ with center $i \in \mathbb{Z}^\ell$ is $|\{j \in \mathbb{Z}^\ell : d(i, j) \leq k\}| = (2k + 1)^\ell$ and so

$$|\{j \in \mathbb{Z}^\ell : d(i, j) = k\}| = (2k + 1)^\ell - (2k - 1)^\ell.$$

Note that if $d(i, j) = 1$ then c_i and c_j their closures have a non-empty intersection. On the other hand given two cubes c_i, c_j with $d(i, j) > 1$ we can shift them simultaneously such that the center of c_i coincides with the origin and closure of the the shifted c_i can be identified with W_a , $a = \frac{1}{2}$, while the shifted c_j is located in the complement of $W_b = [-b, b]^\ell$, $b = d(i, j) - \kappa$ for all $0 < \kappa < \frac{1}{2}$. (This notation corresponds to that one which we used in [6].)

We apply Corollary 1 to all these pairs c_i, c_j with $d(i, j) > 1$ and obtain for arbitrary $\delta > 0$, using (3) and the translation invariance of X ,

$$|\text{Cov}(X_i, X_j)| \leq 2 \left(\mathbb{E}|X(W_{\frac{1}{2}}, Y_t)|^{2+\delta} \right)^{\frac{2}{2+\delta}} \left(\beta \left(\frac{1}{2}, d(i, j) - \kappa \right) \right)^{\frac{\delta}{2+\delta}} \quad (7)$$

For pairs c_i, c_j with $d(i, j) = 1$ we use the inequality

$$|\text{Cov}(X_i, X_j)| \leq \sqrt{\text{Var}(X_i) \cdot \text{Var}(X_j)} = \text{Var}(X_i) = \text{Var}(X(W_{\frac{1}{2}}, Y_t)).$$

By writing \sum_i for the summation over all $i \in \mathbb{Z}^\ell$ (which in our setting is equivalent to summation over all $i \in \mathbb{Z}^\ell \cap [-n, n]^\ell$)

$$\begin{aligned} \text{Var}(X(W_n^o, Y_t)) &= \text{Var}\left(\sum_i X_i\right) \\ &= \sum_i \text{Var}(X_i) + \sum_i \sum_{j: d(i, j)=1} \text{Cov}(X_i, X_j) + \sum_i \sum_{k=2}^{2n-1} \sum_{j: d(i, j)=k} \text{Cov}(X_i, X_j) \\ &\leq \sum_i \text{Var}(X_i) + \sum_i \sum_{j: d(i, j)=1} \text{Var}(X_i) + \sum_i \sum_{k=2}^{2n-1} \sum_{j: d(i, j)=k} 2 \left(\mathbb{E}|X_i|^{2+\delta} \right)^{\frac{2}{2+\delta}} \left(\beta \left(\frac{1}{2}, k - \kappa \right) \right)^{\frac{\delta}{2+\delta}} \\ &\leq (2n)^\ell \text{Var}(X_1) + (2n)^\ell (3^\ell - 1) \text{Var}(X_1) \\ &\quad + 2 \left(\mathbb{E}(|X_1|^{2+\delta}) \right)^{\frac{2}{2+\delta}} (2n)^\ell \sum_{k=2}^{2n-1} ((2k+1)^\ell - (2k-1)^\ell) \left(\beta \left(\frac{1}{2}, k - \kappa \right) \right)^{\frac{\delta}{2+\delta}} \\ &\leq (2n)^\ell \left[3^\ell \text{Var}(X_1) + 2 \left(\mathbb{E}(|X_1|^{2+\delta}) \right)^{\frac{2}{2+\delta}} 2^{2\ell-1} \sum_{k=2}^{2n-1} k^{\ell-1} \left(\beta \left(\frac{1}{2}, k - \kappa \right) \right)^{\frac{\delta}{2+\delta}} \right] \end{aligned}$$

Hence, for $X(W_n^o, Y_t)$ divided by the volume of W_n and using the upper bound for β as given in Theorem 5.3 in [6], we obtain for any $\theta \in (0, 1)$, there is a

constant $\chi(\theta)$ such that

$$\begin{aligned}
& \text{Var} \left(\frac{1}{(2n)^\ell} X(W_n^o, Y_t) \right) \\
& \leq \frac{3^\ell}{(2n)^\ell} \text{Var}(X_1) + 2^\ell (\mathbb{E}|X_1|^{2+\delta})^{\frac{2}{2+\delta}} \frac{1}{(2n)^\ell} \sum_{k=2}^{2n-1} k^{\ell-1} \left(\beta \left(\frac{1}{2}, k - \kappa \right) \right)^{\frac{\delta}{2+\delta}} \\
& \leq \frac{3^\ell}{(2n)^\ell} \text{Var}(X_i) + 2^\ell (\mathbb{E}|X_1|^{2+\delta})^{\frac{2}{2+\delta}} \frac{1}{(2n)^\ell} \chi(\theta) \sum_{k=2}^{2n-1} k^{\ell-1} (k - \kappa)^{-\frac{\theta\delta}{2+\delta}}.
\end{aligned}$$

Let us consider the asymptotic behavior for $n \rightarrow \infty$ of

$$\frac{1}{(2n)^\ell} \sum_{k=2}^{2n-1} k^{\ell-1} (k - \kappa)^{-\frac{\theta\delta}{2+\delta}}.$$

Let us denote $\frac{\theta\delta}{2+\delta} = 1 - \rho$. Note that $k^{\ell-1} (k - \kappa)^{-(1-\rho)}$ is increasing in k if $\ell \geq 2$. Thus

$$\frac{1}{(2n)^\ell} \sum_{k=2}^{2n-1} k^{\ell-1} (k - \kappa)^{-(1-\rho)} \leq \frac{1}{(2n)^\ell} 2n(2n)^{\ell-1} (2n - \kappa)^{-(1-\rho)} = (2n - \kappa)^{-(1-\rho)}.$$

Hence, we have

$$\text{Var} \left(\frac{1}{(2n)^\ell} X(W_n, Y_t) \right) \leq \frac{3^\ell}{(2n)^\ell} \text{Var}(X_1) + 2^\ell (\mathbb{E}|X_1|^{2+\delta})^{\frac{2}{2+\delta}} \chi(\theta) (2n - \kappa)^{-(1-\rho)} \quad (8)$$

and thus

$$\text{Var} \left[\frac{1}{(2n)^\ell} X(W_n, Y_t) \right] \leq O \left(n^{-(1-\rho)} \right), \quad n \rightarrow \infty.$$

which is the assertion (5).

6 Ergodic theorems for subadditive functionals of random tessellations

If the functional is not additive, we cannot use the method used in Section 5. But for subadditive functionals, Theorem 2.9 in [4] by Akcoglu-Krengel and Theorem 2.3 in [4] by Smythe together with the property that STIT tessellations are β -mixing can be applied to formulate an ergodic theorem. Here we do not aim at the highest level of generality, but on a formulation similar to Theorem 2, which appears to be well adapted to applications in stochastic geometry.

Theorem 4 *Let Y_t be the STIT tessellation at time $t > 0$ determined by the hyperplane measure Λ satisfying Assumptions I and II. Further, for any $V \in \mathcal{V}$ let $X(V, \cdot) : \mathbb{T} \wedge V \rightarrow \mathbb{R}$ be a functional with the following properties (briefly we write $X(V, y)$ for $X(V, y \wedge V)$):*

1. If $n \in \mathbb{N}$ and $V_1, \dots, V_n \in \mathcal{V}$ are pairwise disjoint such that $\bigcup_{r=1}^n V_r \in \mathcal{V}$, then

$$X\left(\bigcup_{r=1}^n V_r, \cdot\right) \leq \sum_{r=1}^n X(V_r, \cdot) \quad (\text{subadditivity}). \quad (9)$$

2. $X(V, y) = X(V + i, y + i)$ for all $V \in \mathcal{V}$, $i \in \mathbb{R}^\ell$, $y \in \mathbb{T}$.

3. $\gamma = \inf \left\{ \frac{1}{(2n)^\ell} \mathbb{E}X([-n, n]^\ell, Y_t), n \in \mathbb{N} \right\} > -\infty$.

Then

$$\lim_{n \rightarrow \infty} \frac{1}{(2n)^\ell} X([-n, n]^\ell, Y_t) = \gamma \quad a.s. \quad (10)$$

For the L^1 -convergence in (10) it is sufficient if (9) holds for $n = 2$ only (2-subadditivity).

Note that for additive X we have $\gamma = \mathbb{E}X([0, 1]^\ell, Y_t)$.

Comments on the proof: This is a special case of a subadditive process in the sense of Definition 2.1 in Chapter 6 in [4]; the sequence $(W_n : n \in \mathbb{N})$ is increasing and therefore a regular family, see Definition 2.4 ibidem. As Y_t is β -mixing, it is ergodic. Thus the theorem is an immediate corollary of Theorems 2.9 and 2.3 ibidem.

Examples: Besides the additive functionals described in the example in Section 4, some subadditive functionals which fulfill the assumptions of Theorem 4 are:

1. Sum of the values of translation invariant nonnegative functionals of those cells C of the tessellation which are *completely* contained in a window W_n , i.e. $C \subset W_n$. This can e.g. be the intrinsic volumes of these cells, or the diameter (i.e. the maximal breadth) of these cells, or the number of k -dimensional faces of these cells, $k = 0, 1, \dots, \ell - 1$.
2. As before, but for all cells or parts of cells of a tessellation y_t say, which are visible in the window W_n , i.e. for all $C \in y_t \wedge W_n$.
3. Power of order $0 < \alpha < 1$ of a nonnegative functional, e.g. $(\lambda_{\ell-1}(\partial y_t \wedge W_n))^\alpha$, where $\lambda_{\ell-1}$ denotes the $(\ell - 1)$ -dimensional volume.

In all these examples, the functional X is nonnegative, and hence $\gamma \geq 0$.

7 Discussion

The results in [10] indicate that several second-order quantities of the isotropic STIT tessellations are between the corresponding values for isotropic Poisson hyperplane tessellations on one side and for Poisson-Voronoi tessellations on the other side. Thus one can use results for these two other models to formulate

conjectures concerning the variance of functionals for isotropic STIT tessellations.

Acknowledgments The authors thank Lothar Heinrich for essential suggestions and comments. They are indebted for the support of Program Basal CMM from CONICYT (Chile) and by DAAD (Germany).

References

- [1] DALEY, D.J. AND VERE-JONES, D. (2008). *An Introduction to the Theory of Point Processes. Vol. II: General Theory and Structure*. 2nd ed., Springer.
- [2] HEINRICH, L. (1994). Normal approximation for some mean-value estimates of absolutely regular tessellations. *Math. Methods Statist.* **3**, 1–24.
- [3] HEINRICH, L. (2012). Asymptotic methods in statistics of random point processes. in: *E. Spodarov (ed.): Lectures on Stochastic Geometry, Spatial Statistics and Random Fields. Asymptotic Methods*. Lecture Notes in Mathematics, Springer, 115–150.
- [4] KRENGEL, U. (1985). *Ergodic Theorems*. de Gruyter, Berlin.
- [5] MARTÍNEZ, S. AND NAGEL, W. (2012). Ergodic description of STIT tessellations. *Stochastics: An Int. Journ. of Prob. and Stoch. Proc.* **84**, 113–134.
- [6] MARTÍNEZ, S. AND NAGEL, W. (2016). The β -mixing rate of STIT tessellations. *Stochastics: An Int. Journ. of Prob. and Stoch. Proc.* **88**, 396–414.
- [7] MECKE, J., NAGEL, W. AND WEISS, V. (2008). A global construction of homogeneous random planar tessellations that are stable under iteration. *Stochastics: An Int. Journ. of Prob. and Stoch. Proc.* **80**, 51–67.
- [8] NAGEL, W. AND WEISS, V. (2005). Crack STIT tessellations: Characterization of stationary random tessellations stable with respect to iteration. *Adv. Appl. Probab.* **37**, 859–883.
- [9] NGUYEN, X. X. AND ZESSIN, H. (1979) Ergodic theorems for spatial processes. *Zeitschr. Wahrsch. verw. Geb.* **48**, 133–158.
- [10] REDENBACH, C. AND THÄLE, C. (2011). Second-order comparison of three fundamental tessellation models. *Statistics: A Journal of Theoretical and Applied Statistics.* **47**, 237–257.

- [11] SCHNEIDER, R. AND WEIL, W. (2008). *Stochastic and Integral Geometry*. Springer.
- [12] SCHREIBER, T. AND THÄLE, C. (2010). Second-order properties and central limit theory for the vertex process of iteration infinitely divisible and iteration stable random tessellations in the plane. *Adv. Appl. Probab.* **42**, 913–935.
- [13] SCHREIBER, T. AND THÄLE, C. (2012). Second-order theory for iteration stable tessellations. *Probab. Math. Statist.*, **32**, 281–300.
- [14] SCHREIBER, T. AND THÄLE, C. (2013). Limit theorems for iteration stable tessellations. *The Annals of Probability*, **41**, 2261–2278.
- [15] TEMPEL'MAN, A. A. (1972). Ergodic theorems for general dynamical systems. *Trans. Moscow Math. Soc.* **26**, 94–132.
- [16] THÄLE, C. AND WEISS, V. (2013). The combinatorial structure of spatial STIT tessellations. *Discrete & Combinatorial Geometry* **50**, 649–672.
- [17] VOLKONSKII V.A. AND ROZANOV, YU.A. (1959). Some limit theorems for random functions. I. *Theory Probab. Appl.* **4**, 178–197.
- [18] WEISS, V. AND COWAN, R. (2011). Topological relationships in spatial tessellations. *Adv. Appl. Probab.* **43**, 963–984.
- [19] YOSHIHARA, K.-I. (1976). Limiting behavior of U-statistics for stationary absolutely regular processes. *Zeitschr. Wahrsch. verw. Geb.* **35**, 237–252.